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Kinetic theory for granular flow of dense, slightly inelastic, slightly rough spheres

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A general set of conservation equations and constitutive integrals for the dynamic properties of the rapid flow of a granular material consisting of slightly inelastic and slightly rough spherical particles is derived by following an approach used in the kinetic theory of dense gases. By taking moments of the translational and rotational particle velocities in the general transport moment equation and making the Enskog approximation, the singlet velocity distribution function is determined. As a result, the constitutive relations and coefficients such as stresses, energy fluxes, rates of translational and rotational energy interchanges, shear viscosity, spin viscosity, bulk viscosity and 'thermal' conductivities are obtained. The present theory incorporates the kinetic as well as the collisional contributions for stresses and energy fluxes. Thus, it is appropriate for dilute as well as dense concentrations of solids. For the case of simple shear flow, there is favourable agreement between the theoretical predictions of stresses and both the experimental measurements and the results from computer simulations.

1. Introduction

In recent years, a number of kinetic theories based upon the approaches used in the kinetic theory of dense gases were developed for the rapid flows of granular materials; for example, Savage & Jeffrey (1981), Jenkins & Savage (1983), Ahmadi & Shahinpoor (1983), Lun *et al.* (1984), Jenkins & Richman (1985*a*, *b*), Farrell, Lun & Savage (1986), Lun & Savage (1986, 1987), Nakagawa (1988) and Richman (1989). Other types of microstructural statistical theories based upon averaging techniques somewhat more rudimentary than that of the kinetic theory were also proposed, such as McTique (1978), Ogawa, Umemura & Oshima (1980), Shen & Ackermann (1982), Haff (1983), Hopkins & Shen (1986). Also, computer simulations using idealized granular particles were utilized to study the dynamics of granular flow (Walton 1983; Campbell & Brennen 1985; Campbell & Gong 1986; Walton & Braun 1986; Hopkins & Shen 1988; Werner & Haff 1988). The subject has been reviewed by a number of investigators such as Savage (1984), Richman (1986), Jenkins (1987) and Campbell (1990).

Many common granular materials are frictional as well as inelastic. As a result, particles can rotate as well as translate under rapid rates of deformation. Such particle flow behaviour can be easily observed in many geophysical and industrial granular flows such as rock avalanche and granular chute flow. In many cases the dynamic effects of particle surface friction and rotary inertia can play some very important roles and may not be ignored.

Lun & Savage (1987) developed a kinetic theory for a system of inelastic, rough spherical particles to study the effects of particle surface friction and rotary inertia.

They considered only the case of high bulk solids fractions where the major stress contributions are the collisional ones. However, at low solids concentrations the kinetic stress contributions become dominant. In the present study, their theory is extended to incorporate the kinetic stresses as well as the kinetic and collisional energy fluxes for a system of slightly inelastic and slightly rough spheres. We follow the general framework of the kinetic theory of Dahler & Theodosopulu (1975) and Theodosopulu & Dahler (1974*a*, *b*) developed for a system of perfectly elastic, perfectly rough spheres. Conservation equations, constitutive relations and coefficients are obtained explicitly by taking moments of translational and rotational particle velocities in the transport moment equation. Although the detailed computations of the moment method utilized here are quite involved, its basic concept is rather straightforward and is similar in essence to that used by Lun *et al.* (1984). The case of simple shear flow will be studied and the theoretical predictions of stresses will be compared with the experimental results of Savage & Sayed (1984), Hanes & Inman (1985) and Craig, Buckholz & Domoto (1986), and with the computer simulation results of Campbell (1989) and Walton & Braun (1986).

2. Collisional model

The collisional model proposed by Lun & Savage (1987) is employed in the present study. The same collisional model was used in the kinetic theory for plane flows of dense rough, inelastic circular disks by Jenkins & Richman (1985*b*). Two coefficients, e and β, are used to characterize the collision process; e is the usual coefficient of restitution in the normal direction and β is called the roughness coefficient in the tangential direction.

Consider a collision between two spherical particles 1 and 2 each of diameter σ and having translational velocities $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, respectively. The total relative velocity at contact point $\boldsymbol{g}_{12}$ just prior to the collision is

$$\boldsymbol{g}_{12} = \boldsymbol{c}_{12} - \tfrac{1}{2}\sigma \boldsymbol{k} \times \boldsymbol{\Omega}, \tag{2.1}$$

where $\boldsymbol{c}_{12} = \boldsymbol{c}_1 - \boldsymbol{c}_2$ and $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. During a collision the components of $\boldsymbol{g}_{12}$ are changed such that

$$(\boldsymbol{k}\cdot\boldsymbol{g}'_{12}) = -e(\boldsymbol{k}\cdot\boldsymbol{g}_{12}), \quad (\boldsymbol{k}\times\boldsymbol{g}'_{12}) = -\beta(\boldsymbol{k}\times\boldsymbol{g}_{12}), \tag{2.2a, b}$$

where $\boldsymbol{k}$ is the unit vector along the centreline from particle 1 to particle 2, and primed quantities denote values after the collision.

Nakagawa (1988) attempted to improve the above collisional mode by assuming either rolling or complete slip in the contact zone of colliding disks. As a result, the roughness coefficient can be related to the coefficient of restitution, the friction coefficient and the collision angle by a simple expression. From physical considerations, experimental evidence and theoretical analysis of frictional surface deformations (Goldsmith 1960; Maw, Barber & Fawcett 1976, 1981), it is apparent that the coefficient of restitution depends on the inelasticity in the normal direction and the impact velocity whereas the roughness coefficient depends on the tangential inelasticity, particle surface friction and the impact velocity. According to Maw *et al.* (1976, 1981), the collisional process is much more complex than that proposed by Nakagawa (1988). For example, in realistic collisions there can be no slip, micro-slip or complete slip in the contact zone of the colliding spheres (see also Johnson 1982, 1985).

Lun & Savage (1986) studied the effects of an impact-velocity-dependent coefficient of restitution on stresses developed in a granular material consisting of inelastic, smooth spherical particles undergoing rapid deformation. The coefficient of restitution e was assumed to decrease exponentially with increasing impact velocity. This greatly simplified the collision integrals; however, the computations were still formidable. It is not difficult to realize the complexity involved if the effects of particle surface friction and tangential inelasticity were included as well. For the sake of simplicity in the present treatment, both e and β are regarded as merely constant phenomenological coefficients which have been averaged over particle impact velocity and are appropriate for a particular range of granular temperatures.

Generally speaking, the coefficient of restitution e can have a value in the range of $0 \leqslant e \leqslant 1$ while the roughness coefficient β can have a value in the range of $-1 \leqslant \beta \leqslant 1$ (Lun & Savage 1987). The case of $\beta = -1$ represents the collision of perfectly smooth particles, and increasing values of β represent the increasing degrees of particle surface friction. Cases for which $0 < \beta \leqslant 1$ represent situations in which spin reversal occurs following the collision. The impact experiments of discs colliding with flat surfaces and the theoretical analysis of Maw *et al.* (1976, 1981), as well as the experimental measurements presented by Goldsmith (1960, p. 267) confirm the existence of positive values of β (see also Johnson 1982).

The case of $\beta = 1$ which represents the collision of perfectly elastic, perfectly rough particles has been used as a standard model in the kinetic theories of gases and dense fluids to study the effects of rotary inertia and internal energies of complex molecules in a simple way (Chapman & Cowling 1970; Dahler & Theodosopulu 1975). The case of $\beta = 0$ corresponds to the collision model used in the computer simulations of Campbell (1989) in which the particle surface friction and inelasticity are sufficient to eliminate the post-collisional tangential relative velocities.

Since we are considering the case of slightly inelastic and slightly rough particles such that the rate of energy dissipation of the system is small, the value for e is taken to be close to unity whereas the value for β is taken to be close to 1 or -1. The theory of Lun *et al.* (1984) and the computer simulation of smooth, inelastic spheres performed by Walton & Braun (1986) have shown that for the case of simple shear flow, the stresses predicted by the computer simulations agree quite well with those predicted by the theory even for the case of e being as low as 0.6 and for solids fractions up to 0.5. For concentrations near the closest random packing of the flow system, the assumption of binary collisions breaks down.

Using (2.2*a, b*) in (2.1) the relationships between the pre- and post-collisional velocities can be shown to be

$$\boldsymbol{c}_1' - \boldsymbol{c}_1 = -\eta_2 \boldsymbol{c}_{12} - (\eta_1 - \eta_2)\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{c}_{12}) + \tfrac{1}{2}\eta_2 \sigma \boldsymbol{k}\times\boldsymbol{\Omega}, \tag{2.3}$$

$$\boldsymbol{c}_2' - \boldsymbol{c}_2 = \eta_2 \boldsymbol{c}_{12} + (\eta_1 - \eta_2)\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{c}_{12}) - \tfrac{1}{2}\eta_2 \sigma \boldsymbol{k}\times\boldsymbol{\Omega} \tag{2.4}$$

and

$$\boldsymbol{\omega}_1' - \boldsymbol{\omega}_1 = \boldsymbol{\omega}_2' - \boldsymbol{\omega}_2 = -\frac{2\eta_2}{\sigma K}(\boldsymbol{k}\times\boldsymbol{c}_{12}) + \frac{\eta_2}{K}(\boldsymbol{k}\cdot\boldsymbol{\Omega})\boldsymbol{k} - \frac{\eta_2}{K}\boldsymbol{\Omega}, \tag{2.5}$$

where $\eta_1 = \frac{1}{2}(1+e)$, $\eta_2 = \frac{1}{2}(1+\beta)K/(1+K)$ and $K = 4I/m\sigma^2$ is the non-dimensional parameter of moment of inertia. Parameter K can vary in value from zero, when the mass is concentrated at the centre of the sphere, to $\frac{2}{3}$ when the mass is uniformly distributed over the surface of the sphere. For the case of uniform solid sphere treated here, $K = \frac{2}{5}$.

3. Transport equations and constitutive integrals

Considering a fixed volume element d$\boldsymbol{r}$ centred at $\boldsymbol{r}$, the ensemble average of the single-particle quantity ψ is defined as

$$\langle\psi\rangle = \frac{1}{n}\int \psi f^{(1)}(\boldsymbol{r},\boldsymbol{c},\boldsymbol{\omega};t)\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{\omega}, \tag{3.1}$$

where n is the local number density of the particles and $f^{(1)}(\boldsymbol{r},\boldsymbol{c},\boldsymbol{\omega};t)$ is the usual single-particle velocity distribution function. The rate of change of $\langle\psi\rangle$ can be expressed as (Condiff, Lu & Dahler 1965; Lun *et al.* 1984; Lun & Savage 1987)

$$\frac{\partial}{\partial t}\langle n\psi\rangle = n\langle D\psi\rangle - \nabla\cdot\langle n\boldsymbol{c}\psi\rangle - \nabla\cdot\boldsymbol{\theta}(\psi)+\chi(\psi) \tag{3.2}$$

where $D\psi = \boldsymbol{b}\cdot\partial\psi/\partial\boldsymbol{c}$ and $\boldsymbol{b}$ is the body force per unit mass. The second to last term in (3.2) represents a collisional transfer 'flux' term

$$\boldsymbol{\theta}(\psi) \approx -\tfrac{1}{2}\sigma^3\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_1'-\psi_1)(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\boldsymbol{k}f^{(2)}(\boldsymbol{r}-\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_1,\boldsymbol{\omega}_1;\boldsymbol{r}+\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_2,\boldsymbol{\omega}_2;t)$$
$$\times\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2\,\mathrm{d}\boldsymbol{\omega}_1\,\mathrm{d}\boldsymbol{\omega}_2, \tag{3.3}$$

whereas the last term represents a collisional 'source-like' contribution

$$\chi(\psi) = \tfrac{1}{2}\sigma^2\int_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0}(\psi_1'+\psi_2'-\psi_1-\psi_2)(\boldsymbol{c}_{12}\cdot\boldsymbol{k})f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1,\boldsymbol{\omega}_1;\boldsymbol{r},\boldsymbol{c}_2,\boldsymbol{w}_2;t)$$
$$\times\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2\,\mathrm{d}\boldsymbol{\omega}_1\,\mathrm{d}\boldsymbol{\omega}_2 \tag{3.4}$$

and $f^{(2)}(\boldsymbol{r}-\sigma\boldsymbol{k},\boldsymbol{c}_1,\boldsymbol{\omega}_1;\boldsymbol{r},\boldsymbol{c}_2,\boldsymbol{\omega}_2;t)$ is the pair distribution function.

By taking ψ in (3.2) to be the mass m, linear momentum $m\boldsymbol{c}$, angular momentum, $I\boldsymbol{\omega}$, translational kinetic energy $\frac{1}{2}mc^2$ and rotational kinetic energy $\frac{1}{2}I\omega^2$, we obtain the conservation equations for the field variables, which are the mass, linear momentum, mean particle spin, and particle translational and rotational fluctuation kinetic energies:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{u}, \tag{3.5}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \rho\boldsymbol{b}-\nabla\cdot\boldsymbol{P}, \tag{3.6}$$

$$nI\frac{\mathrm{d}\boldsymbol{\omega}_0}{\mathrm{d}t} = -\nabla\cdot\boldsymbol{L}+\chi(I\boldsymbol{\omega}), \tag{3.7}$$

$$\tfrac{3}{2}\rho\frac{\mathrm{d}T_t}{\mathrm{d}t} = -\boldsymbol{P}:\nabla\boldsymbol{u}-\nabla\cdot\boldsymbol{q}_t-\chi_t \tag{3.8}$$

$$\tfrac{3}{2}\rho\frac{\mathrm{d}T_r}{\mathrm{d}t} = -\boldsymbol{L}:\nabla\boldsymbol{\omega}_0-\nabla\cdot\boldsymbol{q}_r-\chi_r-\boldsymbol{\omega}_0\cdot\chi(I\boldsymbol{\omega}). \tag{3.9}$$

It is worth noting that the mean particle spin is regarded as a field variable in the present analysis. As a result, the above set of mean field equations differs from that given by Jenkins & Richman (1985*b*) for a system of inelastic, rough, circular disks in which the conservation of angular momentum is considered to be a higher moment equation. The issue of the inclusion of mean particle spin as a field variable was discussed at some length by McCoy, Sandler & Dahler (1966).

In the above set of conservation equations $\rho = mn = \nu\rho_p$ is the bulk mass density, ν is the bulk solids fraction defined as the ratio of the volume of solids to the total volume, ρ_p is the mass density of a particle, $\boldsymbol{u} = \langle \boldsymbol{c} \rangle$ is the mean bulk velocity, and $\boldsymbol{\omega}_0 = \langle \boldsymbol{\omega} \rangle$ is the mean particle spin velocity. Furthermore, $\frac{3}{2}mT_t = \frac{1}{2}m\langle C^2 \rangle$ is the mean translational fluctuation kinetic energy where $\boldsymbol{C} = \boldsymbol{c}-\boldsymbol{u}$, and $\frac{3}{2}mT_r = \frac{1}{2}I\langle W^2 \rangle$ is the mean rotational fluctuation kinetic energy where $\boldsymbol{W} = \boldsymbol{\omega}-\boldsymbol{\omega}_0$.

The stress tensor $\boldsymbol{P}$, the angular momentum flux $\boldsymbol{L}$, the translational energy flux $\boldsymbol{q}_t$ and the rotational energy flux $\boldsymbol{q}_r$ are each made up of the sum of a kinetic part and a collisional transfer part denoted by subscripts k and c respectively; thus

$$\boldsymbol{P}_k = \rho\langle \boldsymbol{CC} \rangle, \quad \boldsymbol{P}_c = \theta(m\boldsymbol{C}); \qquad \boldsymbol{L}_k = nI\langle \boldsymbol{CW} \rangle, \quad \boldsymbol{L}_c = \theta(I\boldsymbol{W});$$
$$\boldsymbol{q}_{t_k} = \tfrac{1}{2}\rho\langle C^2\boldsymbol{C} \rangle, \quad \boldsymbol{q}_{t_c} = \theta(\tfrac{1}{2}mC^2); \qquad \boldsymbol{q}_{r_k} = \tfrac{1}{2}nI\langle W^2\boldsymbol{C} \rangle, \quad \boldsymbol{q}_{r_c} = \theta(\tfrac{1}{2}IW^2).$$

The term $\chi(I\boldsymbol{\omega})$ is the rate of collisional transfer of angular momentum due to the difference between the local mean spin and the rate of rotational deformation of the bulk material (Lun & Savage 1987). The rate of translational kinetic energy interchange per unit volume is defined as $\chi_t = -\chi(\frac{1}{2}mc^2)$ and the rate of rotational kinetic energy interchange per unit volume is defined as $\chi_r = -\chi(\frac{1}{2}I\omega^2)$.

4. Moment method

Following the approach of Dahler & Theodosopulu (1975) and Theodosopulu & Dahler (1974*a*, *b*), the singlet distribution function $f^{(1)}$ may be written in the form as

$$f^{(1)} = f^{(0)}(1+\phi), \tag{4.1}$$

where $\phi < 1$ and

$$f^{(0)}(\boldsymbol{r}, \boldsymbol{c}, \boldsymbol{\omega}; t) = \frac{n}{(2\pi T_t)^{\frac{3}{2}}(2\pi mT_r/I)^{\frac{3}{2}}} \exp\left(-\frac{(\boldsymbol{c}-\boldsymbol{u})^2}{2T_t}\right) \exp\left(-\frac{I(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2}{2mT_r}\right) \tag{4.2}$$

is the local equilibrium distribution function. The present $f^{(0)}$ differs slightly from the equilibrium distribution functions used for the case of perfectly elastic and perfectly rough spheres by Chapman & Cowling (1970), Condiff *et al.* (1965) and Theodosopulu & Dahler (1974*a*, *b*). In those cases, energy is conserved and there is equipartition of the mean translational and rotational fluctuation kinetic energies. Here different degrees of energy dissipation due to particle inelasticity and surface friction are considered and in general this kind of equipartition of energy would not be satisfied. The form of (4.2), which is written explicitly in terms of the translational and rotational granular temperatures, allows for varying degrees of energy dissipation and in turn allows for different ratios of translational and rotational fluctuation kinetic energies.

The present analysis is based upon the assumption that the rate of energy dissipation is small; so that the non-dimensional parameter defined by Savage & Jeffrey (1981) as the ratio of the characteristic mean relative shear velocity to the r.m.s. of the particle translational fluctuation velocity, $R_t = (\sigma\, \mathrm{d}u/\mathrm{d}y)/\langle C^2 \rangle^{\frac{1}{2}}$, is small. As a result, the perturbation term ϕ which is basically of the order of R_t would be less than unity. This implies that ω_0 which is of the same order as $\mathrm{d}u/\mathrm{d}y$ is also small, hence we may neglect second- and higher-order terms, such as $(\mathrm{d}u/\mathrm{d}y)^2$ and ω_0^2. As a result, the form of the singlet velocity distribution function from (4.2) can be expressed as

$$f^{(0)}(\boldsymbol{r},\boldsymbol{c},\boldsymbol{\omega};t)=\frac{n(1+I\boldsymbol{\omega}_0\cdot\boldsymbol{\omega}/(mT_r))}{(2\pi T_t)^{\frac{3}{2}}(2\pi mT_r/I)^{\frac{3}{2}}}\exp\left(-\frac{(\boldsymbol{c}-\boldsymbol{u})^2}{2T_t}\right)\exp\left(-\frac{I\omega^2}{2mT_r}\right). \tag{4.3}$$

The perturbation ϕ may be approximated by the trial function (see Dahler & Theodosopulu 1975)

$$\phi=\frac{1}{2\rho T_t^2}\boldsymbol{P}_k^*:\boldsymbol{CC}+\frac{2}{5\rho T_t^2}\boldsymbol{q}_{t_k}\cdot\boldsymbol{C}\left(\frac{C^2}{2T_t}-\tfrac{5}{2}\right)+\frac{2}{3\rho T_t T_r}\boldsymbol{q}_{r_k}\cdot\boldsymbol{C}\left(\frac{IW^2}{2mT_r}-\tfrac{3}{2}\right), \tag{4.4}$$

where $\boldsymbol{P}_k^*$ is the symmetric and traceless kinetic stress tensor defined as

$$\boldsymbol{P}_k^*=\rho\langle[\boldsymbol{CC}]^*\rangle \quad \text{and} \quad [\boldsymbol{CC}]^*=\boldsymbol{CC}-\tfrac{1}{3}C^2\boldsymbol{\delta}.$$

The bracket [ ]* represents the traceless part of the tensor enclosed and $\boldsymbol{\delta}$ is the identity tensor. The complete kinetic stress tensor can be written as

$$\boldsymbol{P}_k=\rho T_t\boldsymbol{\delta}+\boldsymbol{P}_k^*$$

where ρT_t may be called the kinetic normal pressure.

The terms in (4.4), which constitute only the first approximation to the distribution function, are known to provide an adequate description of the transport processes. They correspond to the first terms in a complete infinite series such as the Sonine polynomial. The higher-order terms in the complete series were found to alter by only a few percent the numerical values of various transport coefficients such as shear viscosity and thermal conductivity (Condiff *et al.* 1965; Chapman & Cowling 1970). Similar effects are anticipated for the case of slightly inelastic and slightly rough particles treated here.

The perturbation function ϕ used in the theory of Dahler & Theodosopulu (1975, equation 1) has *four* terms in addition to those given in (4.4). Their first two terms, involving T_t and T_r, were formed to account for the so-called relaxation phenomenon between the rotary and translatory kinetic energies based upon the assumption of equipartition of energy. Since we consider different ratios of translational and rotational temperature where equipartition of energy is not obeyed in general, those two terms are irrelevant in the present context. Further discussion of this aspect can be found in the Appendix. The term which depended on the kinetic angular momentum flux $\boldsymbol{L}_k$ is basically a second-order term in the velocity gradient. This is because $\boldsymbol{L}_k$ is a function of the gradient of mean spin $\boldsymbol{\omega}_0$; however, $\boldsymbol{\omega}_0$ is proportional to the gradient of velocity. To be consistent with the first-order approximation in the velocity gradient treated here, this term is neglected. The last term in their theory, which depended on $(\boldsymbol{\omega}_0\cdot\boldsymbol{\omega})$, is accounted for in the present analysis; however, not in ϕ in (4.4) but in (4.3) instead.

We make the Enskog approximation such that we have

$$f^{(2)}(\boldsymbol{r}-\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_1,\boldsymbol{\omega}_1;\boldsymbol{r}+\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_2,\boldsymbol{\omega}_2;t)\approx g_0(\sigma;\nu)f^{(1)}(\boldsymbol{r}-\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_1,\boldsymbol{\omega}_1;t)f^{(1)}(\boldsymbol{r}+\tfrac{1}{2}\sigma\boldsymbol{k},\boldsymbol{c}_2,\boldsymbol{\omega}_2;t), \tag{4.5}$$

where $g_0(\sigma;\nu)$ is the equilibrium radial distribution function at contact. The radial distribution function proposed by Lun & Savage (1986) is adopted in the present study and it is written as

$$g_0(\nu)=(1-\nu/\nu_m)^{-5\nu_m/2}, \tag{4.6}$$

where ν_m represents the maximum possible solids fraction of the system. For finite granular flow systems such as some of the shear cell tests on polystyrene beads and glass beads conducted by Savage & Sayed (1984) and Hanes & Inman (1985), the values of ν_m can be as low as about 0.55 depending upon the size and geometry of the shear space in the test devices. For the closest random packing of spheres ν_m is about 0.64, while for the closest regular packing ν_m is 0.7404.

The present moment method for determining the singlet distribution function and subsequently the constitutive relations is utilized in the same spirit as some of the well-known analytical methods such as the Ritz–Galerkin integral method for the study of vibrations and the von Kármán momentum integral method for the study of boundary layers, as opposed to the common perturbation methods. Roughly speaking, it is a method whereby an appropriate trial function is assumed and its coefficients are solved by satisfying the corresponding higher moment equations. In the present case, the trial function is ϕ. Once the coefficients are determined through the detailed balance of terms in the linearized moment equations, the trial function essentially represents the best possible linear approximate solution.

The quantities $\boldsymbol{P}_k^*$, $\boldsymbol{q}_{t_k}$ and $\boldsymbol{q}_{r_k}$ in (4.4) can be obtained by satisfying the moment equations which are generated by taking ψ to be the higher moments of particle translational and rotational velocities such as $\frac{1}{2}m[\boldsymbol{cc}]^*$, $\frac{1}{2}mc^2$ and $\frac{1}{2}I\omega^2\boldsymbol{c}$ in (3.2) respectively. By using arguments of tensorial isotropy and flux homogeneity, and retaining only the first-order terms of the mean velocity gradient, the solids concentration gradient, the translational and the rotational temperature gradients, and neglecting the unsteady terms (Chapman & Cowling 1970, p. 120), we obtain a closed set of simultaneous equations which we can solve for $\boldsymbol{P}_k^*$, $\boldsymbol{q}_{t_k}$ and $\boldsymbol{q}_{r_k}$. Thus we find

$$\boldsymbol{P}_k^* = -2\mu_k \boldsymbol{S}, \tag{4.7}$$

$$\boldsymbol{q}_{t_k} = -\frac{\lambda}{g_0 \Gamma_1}[\Gamma_2 \nabla T_t + \Gamma_3 \nabla T_r], \tag{4.8}$$

$$\boldsymbol{q}_{r_k} = -\frac{\lambda}{g_0 \Gamma_1}[\Gamma_4 \nabla T_t + \Gamma_5 \nabla T_r], \tag{4.9}$$

$$\mu_k = \mu\left\{\frac{1+\frac{8}{5}[\eta_1(3\eta_1-2)+\frac{1}{2}\eta_2(6\eta_1-1-2\eta_2)-\eta_2^2 T_r/(KT_t)]\,\nu g_0}{g_0[(2-\eta_1-\eta_2)(\eta_1+\eta_2)+\eta_2^2 T_r/(6KT_t)]}\right\}, \tag{4.10}$$

where $\mu = 5m(T_t/\pi)^{\frac{1}{2}}/16\sigma^2$, $\lambda = 75m(T_t/\pi)^{\frac{1}{2}}/64\sigma^2$ and $\boldsymbol{S} = \frac{1}{2}(u_{i,j}+u_{j,i})-\frac{1}{3}u_{k,k}\delta_{ij}$. The expressions for μ and λ may be identified as the shear viscosity and thermal conductivity for perfectly elastic and smooth particles at dilute concentrations. The symbol μ_k may be called the kinetic shear viscosity for the case of slightly inelastic and slightly rough particles.

Lun *et al.* (1984) presented a term proportional to the gradient of particle number density (i.e. ∇n) in their kinetic energy flux equation for smooth particles. As pointed out by Lun *et al.*, such a term contained factor of $(1-\eta_1)$ which had been assumed to be small and therefore was basically a higher-order term. A recent study of granular shute flows by Johnson, Nott & Jackson (1990) using the theory of Lun *et al.* (1984) found that the ∇n term was indeed smaller than the other terms in the expression. Similar terms were found in the kinetic translational and rotational energy fluxes in (4.8) and (4.9) respectively for rough spheres. For uniformity in the order of terms in all constitutive relations they are not presented.

The functions Γ_i are as follows:

$$\Gamma_1 = \tfrac{1}{8}a_1 a_6 - \tfrac{25}{24}a_2 a_5, \quad \Gamma_2 = \tfrac{2}{5}a_3 a_6 + \tfrac{2}{3}a_2 a_7 \nu g_0, \quad \Gamma_3 = \tfrac{2}{3}a_2 a_8 + \tfrac{2}{5}a_4 a_6 \nu g_0,$$
$$\Gamma_4 = \tfrac{1}{5}a_3 a_5 + \tfrac{1}{25}a_1 a_7 \nu g_0, \quad \Gamma_5 = \tfrac{1}{25}a_1 a_8 + \tfrac{1}{5}a_4 a_5 \nu g_0,$$

where

$$a_1 = \eta_1(41-25\eta_1) - 8(\eta_1+\eta_2)^2 + \eta_2(41-25\eta_2) - \frac{7\eta_2^2 T_r}{KT_t}, \quad a_2 = \frac{\eta_2^2}{K},$$

$$a_3 = \tfrac{5}{2} + \left[6\eta_1^2(4\eta_1-3) - 4\eta_2(2\eta_1 - 4\eta_1\eta_2 + \eta_2) + \frac{8\eta_1\eta_2^2 T_r}{KT_t}\right]\nu g_0, \quad a_4 = \frac{4\eta_2^2}{K}(2\eta_1-1),$$

$$a_5 = \frac{3\eta_2^2}{K} + \left(\frac{\eta_2^2}{K^2} - \frac{\eta_2}{K}\right)\frac{T_r}{T_t}, \quad a_6 = \eta_2\left(1 + \frac{7}{3K} - \frac{2\eta_2}{K} - \frac{\eta_2}{K^2}\right) + \eta_1\left(1 - \frac{4\eta_2}{3K}\right),$$

$$a_7 = \frac{4\eta_2^2}{K}(2\eta_1-1) + 8\eta_1\left[\frac{\eta_2^2}{K} + \left(\frac{\eta_2^2}{K^2} - \frac{\eta_2}{K}\right)\frac{T_r}{T_t}\right], \quad a_8 = \tfrac{3}{2} + \frac{4\eta_2^2}{K^2}(2\eta_1-1)\,\nu g_0.$$

Using (4.1)–(4.9) in (3.4) and (3.5), we obtain the various constitutive relationships as defined earlier in §3. The kinetic and collisional contributions to the total stress tensor are

$$\boldsymbol{P}_k = \rho T_t \boldsymbol{\delta} - 2\mu_k \boldsymbol{S}, \tag{4.11}$$

$$\boldsymbol{P}_c = 4\eta_1 \nu\rho g_0 T_t \boldsymbol{\delta} - \tfrac{8}{5}(2\eta_1+3\eta_2)\mu_k \nu g_0 \boldsymbol{S} - \frac{256}{5\pi}\mu\nu^2 g_0[\eta_1 \nabla\cdot\boldsymbol{u}\boldsymbol{\delta} + \tfrac{3}{10}(4\eta_1+3\eta_2)\boldsymbol{S}]$$
$$- \frac{192}{5\pi}\eta_2 \mu\nu^2 g_0 \boldsymbol{\delta}\times(2\boldsymbol{\omega}_0 - \nabla\times\boldsymbol{u}). \tag{4.12}$$

The total stress tensor may be written as

$$\boldsymbol{P} = [\rho T_t(1+4\eta_1\nu g_0) - \mu_b \nabla\cdot\boldsymbol{u}]\boldsymbol{\delta} - 2\mu_s \boldsymbol{S} - \zeta\boldsymbol{\delta}\times(2\boldsymbol{\omega}_0 - \nabla\times\boldsymbol{u}). \tag{4.13}$$

The bulk viscosity μ_b, shear viscosity μ_s and the spin viscosity ζ are given as

$$\mu_b = \frac{256}{5\pi}\eta_1 \mu\nu^2 g_0, \tag{4.14}$$

$$\mu_s = \mu_k[1 + \tfrac{4}{5}(2\eta_1+3\eta_2)\nu g_0] + \frac{3(4\eta_1+3\eta_2)}{20\eta_1}\mu_b, \tag{4.15}$$

$$\zeta = \frac{192}{5\pi}\eta_2 \mu\nu^2 g_0. \tag{4.16}$$

In general the total stress tensor is anisotropic because rough particles with rotary inertia are being considered.

The kinetic translational energy flux $\boldsymbol{q}_{t_k}$ is obtained previously in (4.8) and the collisional flux of translational energy $\boldsymbol{q}_{t_c}$ is found to be

$$\boldsymbol{q}_{t_c} = \tfrac{12}{5}(\eta_1 + \tfrac{2}{3}\eta_2)\nu g_0 \boldsymbol{q}_{t_k} - \frac{4}{\pi^{\frac{1}{2}}}(\eta_1+\eta_2)\rho\nu g_0 \sigma T_t^{\frac{1}{2}} \nabla T_t. \tag{4.17}$$

Therefore the total translational energy flux may be written as

$$\boldsymbol{q}_t = [1 + \tfrac{12}{5}(\eta_1 + \tfrac{2}{3}\eta_2)\nu g_0]\boldsymbol{q}_{t_k} - \frac{4}{\pi^{\frac{1}{2}}}(\eta_1+\eta_2)\rho\nu g_0 \sigma T_t^{\frac{1}{2}} \nabla T_t. \tag{4.18}$$

Similarly, the kinetic rotational energy flux $\boldsymbol{q}_{r_k}$ is given in (4.9) while the collisional flux of rotational energy $\boldsymbol{q}_{r_c}$ is

$$\boldsymbol{q}_{r_c} = \frac{8\eta_2}{3K}\nu g_0 \boldsymbol{q}_{r_k} - \frac{4\eta_2}{K}\rho\nu g_0 \sigma \nabla T_r. \tag{4.19}$$

As a result, the total rotational energy flux is

$$\boldsymbol{q}_r = \left(1+\frac{8\eta_2}{3K}\nu g_0\right)\boldsymbol{q}_{r_k} - \frac{4\eta_2}{K}\rho\nu g_0 \sigma \nabla T_r. \tag{4.20}$$

The rate of collisional transfer of angular momentum is expressed as

$$\chi(I\boldsymbol{\omega}) = -2\zeta(2\boldsymbol{\omega}_0 - \nabla\times\boldsymbol{u}). \tag{4.21}$$

The kinetic and collisional angular momentum fluxes, $\boldsymbol{L}_k$ and $\boldsymbol{L}_c$, are found to be zero.

The present analysis has been based upon the assumption of small R_t which implies small inelasticity and roughness. The expression for the collisional rate of translational kinetic energy interchange per unit volume carried out to the same order of approximation as the expressions for stress and energy fluxes is

$$\chi_t = \frac{48\rho_p \nu^2 g_0}{\pi^{\frac{1}{2}}\sigma} T_t^{\frac{1}{2}}\left\{[\eta_1(1-\eta_1)+\eta_2(1-\eta_2)]\,T_t - \frac{\eta_2^2}{K}T_r\right\}. \tag{4.22}$$

Similarly, the expression for the collisional rate of rotational kinetic energy interchange per unit volume is

$$\chi_r = -\frac{48\rho_p \nu^2 g_0 \eta_2}{\pi^{\frac{1}{2}}\sigma K} T_t^{\frac{1}{2}}\left[\eta_2 T_t - \left(1-\frac{\eta_2}{K}\right)T_r\right]. \tag{4.23}$$

Terms such as $(1-\eta_1)\nabla\cdot\boldsymbol{u}$ were neglected in (4.22) and (4.23), for they are basically of higher order than the leading terms. Note that χ_t and χ_r incorporate not only the energy dissipation from inelasticity and surface friction associated with the coefficients e and β but also the possible exchange between the translational and rotational kinetic energies.

By taking $\beta = -1$, i.e. $\eta_2 = 0$, in (4.7)–(4.23) the results of Lun *et al.* (1984) for *slightly inelastic*, *perfectly smooth* particles are recovered. If we, further, take $e = 1$ corresponding to *perfectly elastic* and *perfectly smooth* particles, the present results reduce to the classical results of the first-order approximation from the dense smooth hard-sphere kinetic theory (see Davis 1973; Chapman & Cowling 1970). If we consider the limits of $e = 1$ and $\beta = 1$ corresponding to *perfectly elastic* and *perfectly rough* particles, (4.7)–(4.23) reduce to most of the results obtained by Theodosopulu & Dahler (1974*b*) except for the bulk viscosity μ_b and the collisional flux of translational energy $\boldsymbol{q}_{t_c}$. A discussion of the discrepancies is presented in the Appendix.

5. Simple shear flow

We now consider the case of a simple shear flow of $\boldsymbol{u} = u(y)\,\boldsymbol{e}_x$ having uniform ρ, T_t, T_r and constant shear rate $\mathrm{d}u/\mathrm{d}y$. Under these conditions, the mean spin equation (3.7) and the rotational fluctuation kinetic energy equation (3.9) reduce to

$$\chi(I\boldsymbol{\omega}) = 0 \quad \text{and} \quad \chi_r = 0. \tag{5.1a, b}$$

From (4.23) and (5.1*b*), we find that the ratio of the mean rotational to translational fluctuation kinetic energy is given as

$$\frac{T_r}{T_t} = \frac{\eta_2}{(1-\eta_2 K^{-1})}. \tag{5.2}$$

The same temperature ratio was obtained by Lun & Savage (1987) and is plotted here as a function of the roughness coefficient β in figure 1. For $\beta = -1$, the particles are perfectly smooth and all the fluctuation energy is in the translational mode. The frictional dissipation and the energy exchange between the rotational and translational modes increases with increasing β. For $\beta = 1$, the particles are perfectly rough, perfectly elastic and no energy dissipation occurs. As a result, there is equipartition of fluctuation kinetic energy between the rotary and translatory modes.

The translational fluctuation kinetic energy (3.8) reduces to a balance between the shear work and the rate of translational kinetic energy interchange:

$$P_{xy}\frac{\mathrm{d}u}{\mathrm{d}y} + \chi_t = 0. \tag{5.3}$$

Energy is supplied from the mean flow to maintain the translational and rotational velocity fluctuations.

From (4.21) and (5.1*a*), the mean spin is found to be equal to the rate of rotational bulk deformation:

$$\boldsymbol{\omega}_0 = \tfrac{1}{2}\nabla \times \boldsymbol{u}.$$

As a result, the stress tensor in (4.13) remains symmetric in this particular case and

$$\boldsymbol{P} = \rho T_t(1+4\eta_1 \nu g_0)\,\boldsymbol{\delta} - \mu_s \frac{\mathrm{d}u}{\mathrm{d}y}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.4}$$

where $P_{xx} = P_{yy} = P_{zz}$ and $P_{xy} = P_{yx}$. From (5.3), (5.4) and (4.22), we obtain the expressions for the non-dimensional shear stress and normal stress, and the parameter R_t:

$$\frac{|P_{xy}|}{\rho_p \sigma^2 (\mathrm{d}u/\mathrm{d}y)^2} = \frac{5}{96}\left(\frac{\pi}{3}\right)^{\frac{1}{2}} \frac{F}{R_t}, \tag{5.5}$$

$$\frac{P_{yy}}{\rho_p \sigma^2 (\mathrm{d}u/\mathrm{d}y)^2} = \frac{\nu(1+4\eta_1 \nu g_0)}{3R_t^2}, \tag{5.6}$$

$$R_t = \left(\frac{1536G}{5\pi F}\right)^{\frac{1}{2}}, \tag{5.7}$$

where

$$F = \frac{\{1+\frac{8}{5}[\eta_1(3\eta_1-2)+\frac{1}{2}\eta_2(6\eta_1-1-2\eta_2)-\eta_2^2 T_r/KT_t]\,\nu g_0\}\,[1+\frac{4}{5}(2\eta_1+3\eta_2)\,\nu g_0]}{g_0[(2-\eta_1-\eta_2)(\eta_1+\eta_2)+\eta_2^2\,T_r/6KT_t]}$$
$$+\frac{192(4\eta_1+3\eta_2)}{25\pi}\nu^2 g_0$$

and $$G = \nu^2 g_0\{[\eta_1(1-\eta_1)+\eta_2(1-\eta_2)]-\eta_2^3(K-\eta_2)^{-1}\}.$$

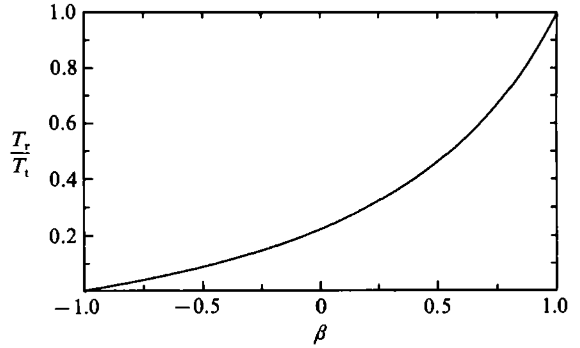

FIGURE 1. Variation of specific kinetic energy ratios, T_r/T_t, with roughness coefficient β.

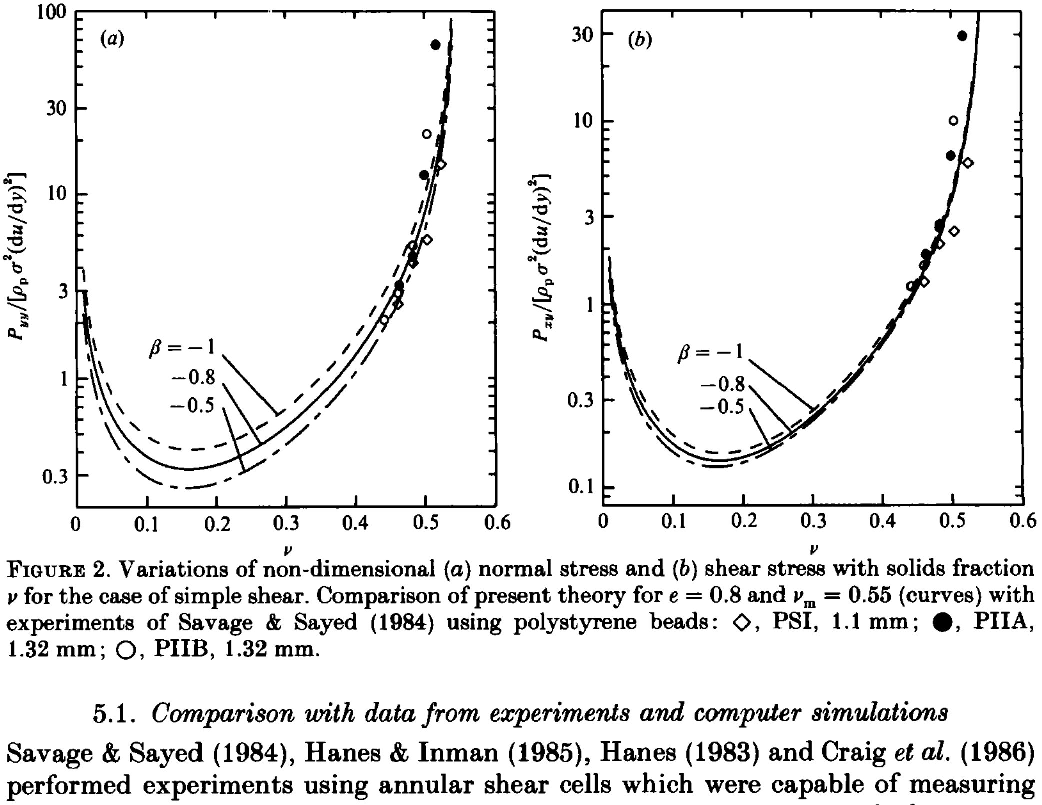

FIGURE 2. Variations of non-dimensional (*a*) normal stress and (*b*) shear stress with solids fraction ν for the case of simple shear. Comparison of present theory for $e = 0.8$ and $\nu_m = 0.55$ (curves) with experiments of Savage & Sayed (1984) using polystyrene beads: ◇, PSI, 1.1 mm; ●, PIIA, 1.32 mm; ○, PIIB, 1.32 mm.

5.1. *Comparison with data from experiments and computer simulations*

Savage & Sayed (1984), Hanes & Inman (1985), Hanes (1983) and Craig *et al.* (1986) performed experiments using annular shear cells which were capable of measuring shear and normal stresses as functions of solids concentration and shear rate. Different boundary conditions were used in the shear cell devices. Savage & Sayed firmly attached very coarse sandpaper having sand grains approximately the same size as the test particles to the top and bottom boundaries of the annular shear region. Hanes & Inman (1985) and Craig *et al.* (1986) glued particles the same as the test particles onto the top and bottom boundaries of the shear zone. Dry spherical glass beads, polystyrene beads and carbon steel spheres were among the materials tested.

The experimental data for stresses employed in the following comparisons are those measurements which were recorded at the highest shear rates investigated for each particular solids fraction of material tested. This serves to minimize the influence of quasi-static effects which can occur at low shearing rates of bulk

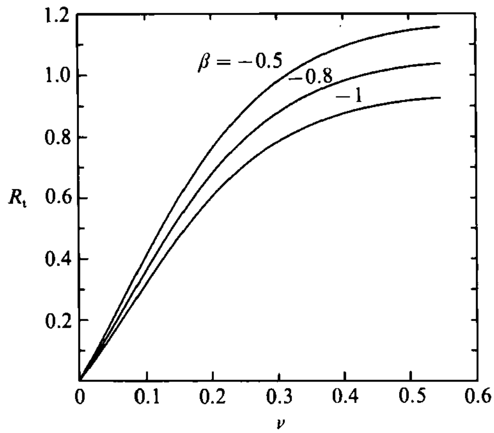

FIGURE 3. Variations of ratio of characteristic mean shear velocity to fluctuation velocity R_t with roughness coefficient β for the case of simple shear where $e = 0.8$ and $\nu_m = 0.55$.

materials. As a consequent, data which may best represent the shearing of bulk material in the grain inertia regime are used to compare with the predictions of the present kinetic theory for rapid flow. Measurements of the normal stress applied at the top boundary of the annular shear zone are also used for the comparisons.

First, the stresses predicted by the present analysis are compared with those obtained by Savage & Sayed (1984) in shearing polystyrene beads in tests PS18-21 and P23-31. The experimental results are shown in figure 2 where the non-dimensional normal and shear stresses are plotted versus the solids fraction ν. The solids fraction for maximum packing of polystyrene beads in those tests was found to be about 0.55. Thus, we set $\nu_m = 0.55$ in (4.6). Little information about the values of e and β for both the glass beads and polystyrene beads is available. A value of $e = 0.8$ is probably in the appropriate range for the polystyrene beads. Calculations using the present theory for cases of $\beta = -1$, -0.8 and -0.5 are shown in the figure. In general, there is substantial agreement between the predictions and the experimental results, especially at solids fractions less than 0.5. Although the polystyrene beads may be relatively smooth to start with, the coarse sandpaper on both the top and bottom boundaries used by Savage & Sayed would probably cause microfractures on the particle surface thus resulting in an increase of individual particle surface friction.

Figure 3 shows the variation of the parameter R_t with ν for the comparisons considered above. The curves are similar to those predicted by the theory of Lun *et al.* (1984) and those obtained by Campbell & Brennen (1985) in their computer simulations of idealized disk-like granular materials. The parameter R_t increases with increasing ν. Even though most kinetic theories assume R_t to be small, for computer simulations and most realistic granular flow systems with high concentrations of solids and relatively low values of e, R_t can be of order one. The curves also show an increase in R_t with increasing β. This is anticipated since an increase in roughness coefficient β represents a decrease in translational temperature due to corresponding increases in rotational temperature and energy dissipation caused by friction and tangential inelasticity.

Next, the stresses measured by Savage & Sayed (1984) and by Hanes & Inman (1985) in shearing dry glass beads are compared with those predicted by the theory. The mean diameter of the glass beads tested by Savage & Sayed was 1.8 mm while the ones tested by Hanes & Inman were 1.85 mm and 1.1 mm. The experimental

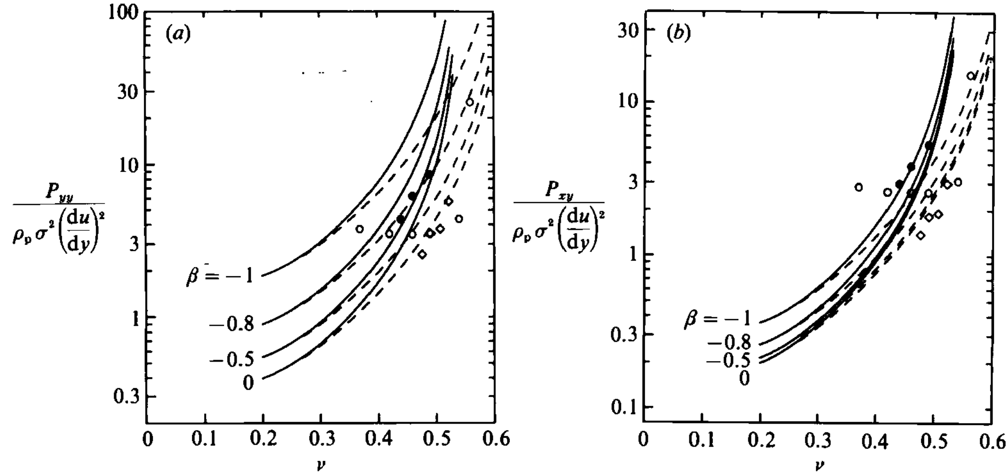

FIGURE 4. Variations of non-dimensional (*a*) normal stress and (*b*) shear stress with solids fraction ν for the case of simple shear. Comparison of present theory for $e = 0.95$ (——, $\nu_m = 0.55$; ----, $\nu_m = 0.64$) with experiments on Ballotini glass beads (Hanes & Inman 1985: ●, 1.85 mm; ○, 1.1 mm; Savage & Sayed 1984: ◇, 1.8 mm).

results are shown in figure 4. The maximum solids fraction for closest packing of 1.8 mm and 1.85 mm glass beads in their respective shear cells was found to be about 0.55 whereas that for the 1.1 mm beads was 0.64. Lun & Savage (1986) conducted experiments to measure the coefficient of restitution for glass beads with diameters ranging from 2.0 to 2.5 mm. The mean coefficient of restitution was found to be about 0.95 over a range of particle impact velocities. This value of e is consistent with the data presented by Goldsmith (1960). According to the estimation made by Lun & Savage (1986), the range of impact velocities experienced by the particles in the shear cell experiments of Savage & Sayed (1984) and Hanes & Inman (1985) is about the same as that measured in the experiments on the coefficient of restitution. Note the misprint of ν in the mean particle impact velocity given by Lun & Savage (1986); for the case of simple shear the expression should be written as

$$\bar{V}_i = \frac{24}{\pi^{\frac{1}{2}}}\nu g_0[(\nu_m/\nu)^{\frac{1}{3}} - 1]\, T^{\frac{1}{2}}_t.$$

Several representative values of β are used in the simple shear calculations; i.e. $\beta = -1$, -0.8, -0.5 and 0. The solid curves were calculated using $\nu_m = 0.55$ and should be compared with the data from testing 1.8 mm and 1.85 mm mean diameter glass beads. The experimental results for the 1.85 mm glass beads fall within the range of stresses as predicted by the theory whereas those for the 1.8 mm glass beads are found to be much lower than both. As pointed out previously by Lun *et al.* (1984), glass beads are brittle; when they are sheared at high shear rates under high loads, the surface of each bead is gradually roughened as a result of the multitude of collisions it experiences and the minute fractures that occur. Thus, one would expect the measured stresses to be lower than those of smooth particles. According to the present comparisons, the stresses measured in shearing 1.8 mm mean diameter glass beads are even lower than those predicted for the case of $\beta = 0$. In view of this, such low values of stresses measured were probably caused not only by high particle surface friction alone but, additionally, partly by slip at the top and bottom boundaries in the annular shear region.

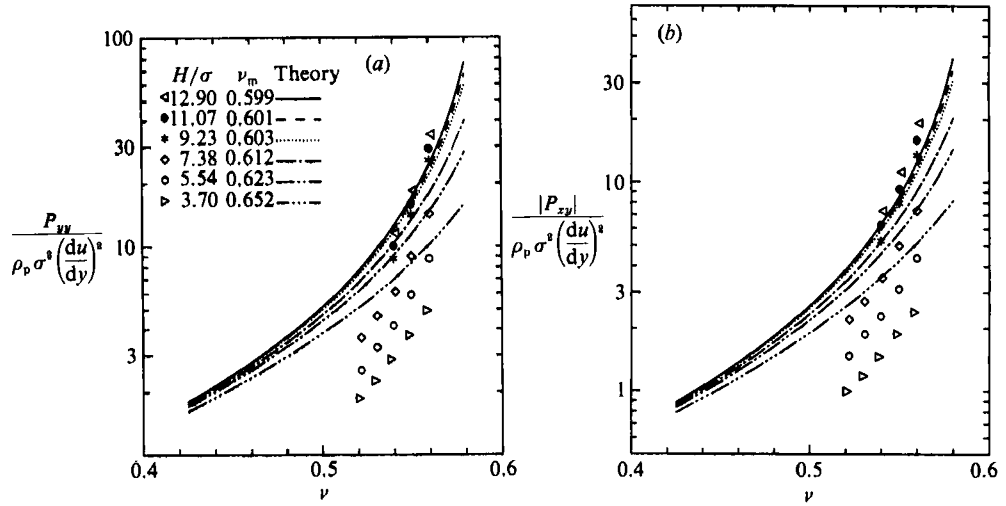

FIGURE 5. Non-dimensional (*a*) normal stress and (*b*) shear stress versus solids fraction ν for the case of simple shear. Comparison of present theory for $e = 0.88$ and $\beta = -0.5$ (curves) with experiments on shearing carbon steel spheres (Craig *et al.* 1986).

The stresses measured in shearing 1.1 mm mean diameter glass beads show a variation with ν which is quite different from the predictions (i.e. the dashed curves for $\nu_m = 0.64$) and the other measurements for larger beads. It is unclear what the cause of such a peculiar variation in the stresses would be.

Figure 5 compares the theoretical predictions with the experimental measurements obtained by Craig *et al.* (1986) in shearing carbon steel spheres at different shear rates, solids concentrations and shear zone thickness, H. The mean values of the non-dimensional shear layer thickness, H/σ, are shown in the figures. The solids fractions at maximum closest packing were measured for the different H tested. Unfortunately the coefficient of restitution of the particles was not determined. Nonetheless, by estimating the mean particle impact velocity for the range of the shear rate tested and inferring from experimental results presented by Goldsmith (1960), the mean coefficient of restitution is found to be about 0.88. A value of -0.5 is used for β.

In figure 5, it is interesting to note that the stresses measured in the tests using a relatively large shear layer thickness H are higher than those for small H, and furthermore they are closer to the theoretical predictions. One should note that at high solids concentrations such as those tested in the experiments, the particles would probably experience simultaneous multiple collisions and frictional rubbing which the present theory has neglected. Savage (1991) found a similar reduction in stresses in his recent computer simulation study of the effects of shear layer thickness on stresses developed in the Couette flow of an assembly of idealized smooth spheres. The decrease in stresses with decreasing H found by Craig *et al.* (1986) could be caused by a number of factors such as slip at the top and bottom solid boundaries, increase in correlations between the particle velocities and the layering effects of particles.

Campbell (1989) used a computer simulation program to obtain the stresses developed in simple shear flow of idealized spherical particles. Moving periodic image cells were applied at the top and bottom boundaries while stationary periodic image cells were used for all the side boundaries of the control volume. In effect, the program may be viewed as a simulation of an infinite medium of bulk materials

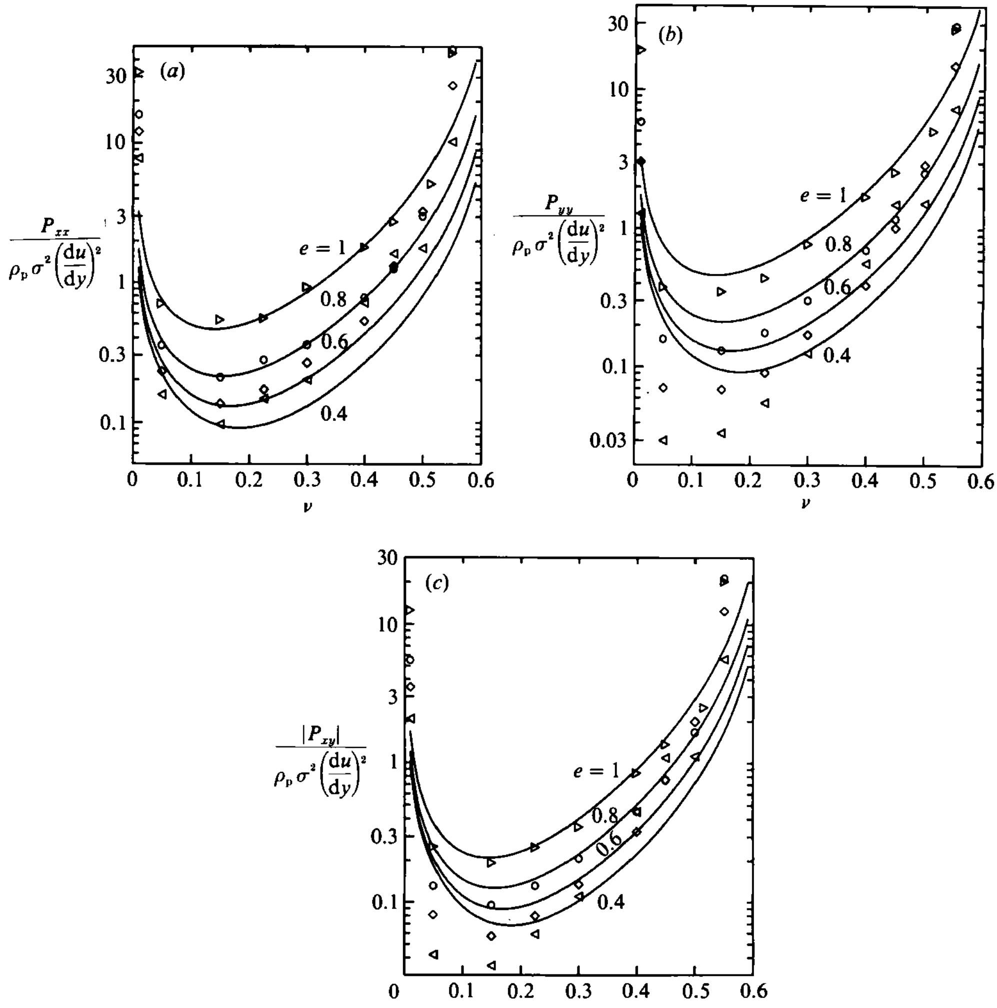

FIGURE 6. Non-dimensional (*a*, *b*) normal stresses and (*c*) shear stress versus solids fraction ν for the cases of simple shear. Comparison of present theory (solid curves) for $\beta = 0$ and $\nu_m = 0.64$ with computer simulations of Campbell (1989) at: ▷, $e = 1$; ○, $e = 0.8$; ◇, $e = 0.6$; ◁, $e = 0.4$.

undergoing simple shearing flow and the control volume represents only a small region within the shear zone. For this reason, a value of $\nu_m = 0.64$ is used for the present calculations. The collision model used by Campbell (1989) is that the particle surface friction and inelasticity are sufficient to eliminate the post-collisional tangential relative velocities. As discussed earlier, this model corresponds to the case of $\beta = 0$ in the present context.

Unlike the physical experiments of Savage & Sayed (1984), Hanes & Inman (1985) and Craig *et al.* (1986) where only one component of the normal stresses, namely P_{yy}, was measured, the computer simulation program can obtain all three (i.e. P_{xx}, P_{yy}, P_{zz}). The results of Campbell (1989) and Walton & Braun (1986) indicated that the normal stresses were anisotropic while the present theory predicts isotropic stresses. The severity of the anisotropy between different components of the normal stresses was found to decrease with increasing ν and increasing e. The component P_{xx} plotted in figure 6(*a*) is the largest normal stress among the three. The component P_{yy} is shown in figure 6(*b*). The comparisons of the theoretical predictions with the

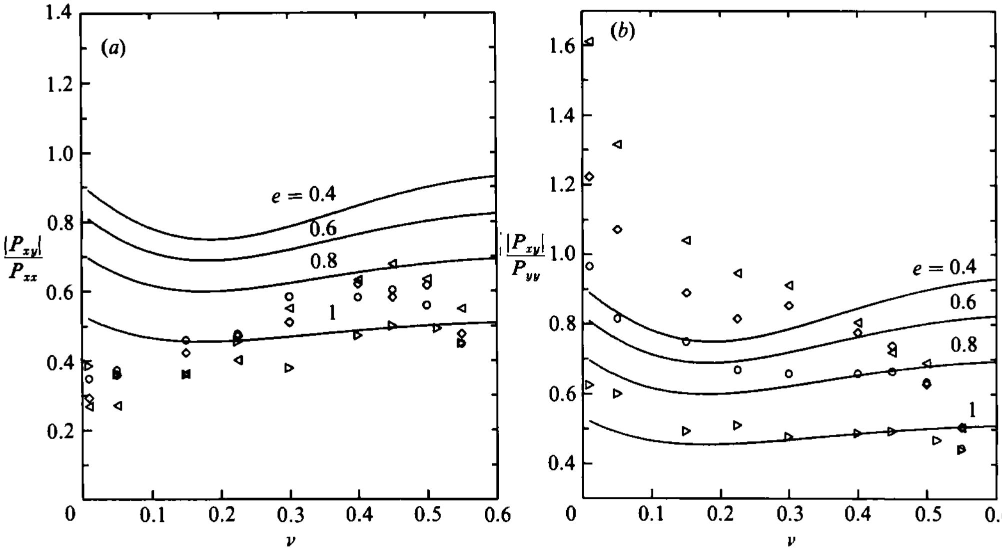

FIGURE 7. Ratio of shear to normal stress, (*a*) P_{xx} and (*b*) P_{yy}, versus solids fraction ν for the case of simple shear. Comparison of present theory (solid curves) for $\beta = 0$ and $\nu_m = 0.64$ with computer simulations of Campbell (1989) at: ▷, $e = 1$; ○, $e = 0.8$; ◇, $e = 0.6$; ◁, $e = 0.4$.

numerical results for P_{zz} differ only slightly from those of P_{yy} and hence are not presented here. The variation of the shear stress P_{xy} with ν is shown in figure 6(*c*).

Although the present theory is expected to be best suited to cases of β close to -1, it is interesting to see that in general there is good agreement between the predictions and a number of simulation results, as shown in figure 6. At a solids fraction of 0.01, the stresses obtained by Campbell (1989) are much higher than those predicted by the theory; the present predictions for $\nu = 0.01$ are shown as the end points of the curves. In the high-solids-fraction region, there exist a few puzzling irregularities in the simulation results. For example, at $\nu = 0.45$ the shear and normal stresses obtained for $e = 0.4$ are *higher* than those of $e = 0.6$ and 0.8. Similar anomalies can also be found at solids fractions such as $\nu = 0.4$, 0.5 and 0.55. Such irregularities are not seen in the present theoretical predictions. The reason for the discrepancies is not clear at the present time.

Figures 7(*a*) and 7(*b*) show respectively the variations of the ratios of shear stress to normal stress, namely $|P_{xy}|/P_{xx}$ and $|P_{xy}|/P_{yy}$, with ν. The simulation results for the two ratios exhibit significant differences which were basically caused by the anisotropies in the normal stresses or in other words the temperatures. This is for the same reason as for the discrepancies between the numerical results and the present theory which assumes an isotropic temperature distribution.

Next, the theoretical predictions are compared with the computer simulation results obtained by Walton & Braun (1986) as well as those of Campbell for $e = 0.8$. Walton & Braun used perfectly smooth (i.e. $\beta = -1$) spherical particles in the simulation program. The data employed here are the results for a constant shear rate of 10 s^{-1}. For brevity, the results for the components P_{yy} and P_{zz} obtained by Walton & Braun were not shown here. They behave more or less like those of Campbell (1989). In general, there is reasonable agreement between the predicted stresses and the results of Walton & Braun, as shown in figure 8. At relatively high concentrations of solids, the stresses obtained by Walton & Braun are somewhat lower than the predictions. The reason for the low values of stresses may be that Walton & Braun

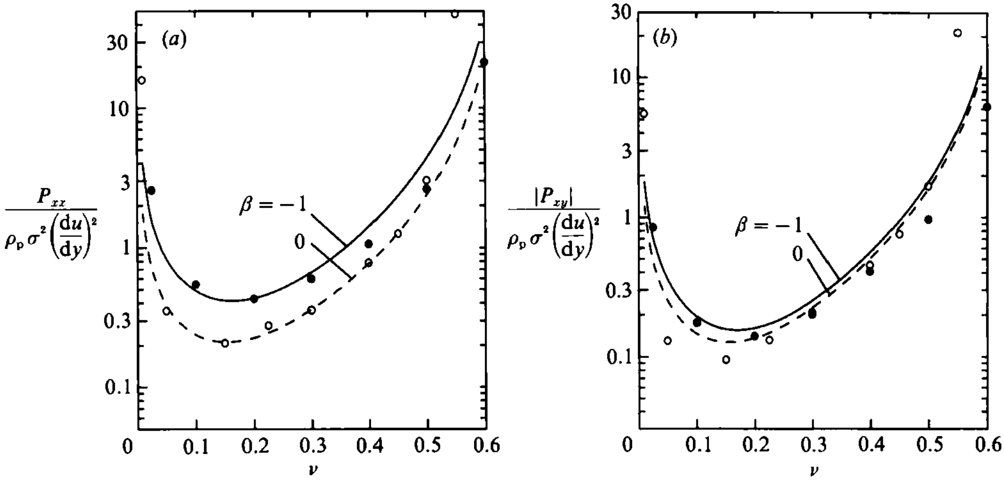

FIGURE 8. Non-dimensional (*a*) normal stress and (*b*) shear stress versus solids fraction ν for the case of simple shear. Comparison of present theory ($\nu_m = 0.64$) (curves) with computer simulations for $e = 0.8$: ●, Walton & Braun (1986), $\beta = -1$; ○, Campbell (1989), $\beta = 0$.

used an impact-time-dependent soft collision model. As a result, the non-dimensional stresses at a particular solids concentration could vary with the shear rate. Such a phenomenon is evident in some other simulations done by Walton & Braun using different shear rates.

6. Conclusion

The present analysis extends the kinetic theory for rapid granular flow of Lun & Savage (1987) to incorporate the kinetic contributions for stresses as well as the kinetic and collisional energy fluxes for the case of slightly inelastic, slightly rough spherical particles. For general flow problems, knowledge of the energy fluxes is especially important. The theory is appropriate for dilute as well as dense concentrations of solids. The case of simple shear flow is examined here in detail. Generally speaking, there is favourable agreement between the theoretical predictions of stresses and the experimental measurements, as well as the computer simulation results. The increase in dissipation due to particle surface friction and tangential inelasticity cause the translational granular temperature to decrease, hence lower stresses are incurred.

One area to which the present theory may be extended is the anisotropic stress phenomenon. For simple shear flow, the theory predicts isotropic stresses whereas the numerical simulation results of Campbell (1989) and Walton & Braun (1986) indicate that the stresses are anisotropic. It is likely that the singlet velocity distribution function with an anisotropic partition of granular temperatures will be required in the analysis.

With proper boundary conditions, the present theory can be applied to solve a number of interesting problems such as granular chute flows under the influence of gravity. The question of boundary conditions next to a solid boundary and a free surface requires further investigation.

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Appendix. Discussion of the results for bulk viscosity and collisional flux of translational energy

For a system of perfectly elastic, perfectly rough dense hard-sphere gases, Theodosopulu & Dahler (1974*b*) obtained the following result for the bulk (or volume) viscosity:

$$\mu_b = \frac{1}{32}\frac{(1+4\nu g_0)^2}{\sigma^2 g_0}\left(\frac{mkT}{\pi}\right)^{\frac{1}{2}}\frac{(K+1)^2}{K}+\frac{16}{\pi\sigma^2}\left(\frac{mkT}{\pi}\right)^{\frac{1}{2}}\nu^2 g_0, \tag{A 1}$$

where $T = \frac{1}{2}(T_t+T_r)$. The last term in (A 1) (which will be referred to as the volumetric term) corresponds exactly to the present result for bulk viscosity given in (4.14) for $e = 1$. The only difference between (A 1) and (4.14) is simply the presence of the first term in (A 1). It is worth noting that this term (which will be referred to as the relaxation term) was derived basing upon the *a priori* assumption of equipartition of energy and from the first two terms involving T_t and T_r in the perturbation function ϕ used by Dahler & Theodosopulu (1975, equation 1), Theodosopulu & Dahler (1974*b*).

For a system of dilute rough-sphere gases, the volumetric term drops out and (A 1) reduces to the familiar result (Chapman & Cowling 1970) as $\nu \rightarrow 0$, $g_0 \rightarrow 1$,

$$\mu_b = \frac{1}{32\sigma^2}(mkT/\pi)^{\frac{1}{2}}(K+1)^2/K \tag{A 2}$$

The stresses in this case can then be expressed as

$$\boldsymbol{P} = (\rho T - \mu_b \nabla\cdot\boldsymbol{u})\boldsymbol{\delta} - 2\mu\boldsymbol{S} \tag{A 3}$$

The bulk viscosity is often interpreted as a relaxation phenomenon because of its origin in the kinetic theory of dilute rough-sphere gases as discussed in Chapman & Cowling (1970). According to Chapman & Cowling μ_b arises because in an expansion or contraction the work done by the pressure alters the translatory energy immediately, but affects the internal energy (such as rotational energy) only after a certain time-lag, through inelastic collisions. The bulk viscosity is found to be proportional to the relaxation time between T_t and T_r.

One peculiarity of (A 2) is that as $K \rightarrow 0$, $\mu_b \rightarrow \infty$ and so does the normal pressure in (A 3). At dilute concentrations where the actual gas molecules may be regarded as point masses and collisions between particles are relatively infrequent, there exists no obvious means by which the pressure can be larger than usual for the case of K being small. This is one of the defects of the rough-sphere model. Further discussion on this can be found in Chapman & Cowling.

Owing to the peculiar behaviour of the relaxation term, the bulk viscosity given in (A 1) does not reduce to the classical result for the case of dense *smooth* hard-sphere gases, whereas the other constitutive relations derived by Theodosopulu & Dahler (1974*b*) (except their collisional flux of translational energy) and all of those obtained in the present theory do. Their collisional flux of translational energy will be discussed shortly. Note that in order to reduce the results of Theodosopulu & Dahler to the smooth-sphere case, one must first express the total temperature T in terms of T_t and T_r accordingly in the relationships.

According to the kinetic theory of dense smooth hard-sphere gases (Chapman & Cowling 1970, p. 306), the bulk viscosity basically originates from the fact that gas particles are finite. Especially in dense concentrations, the finite size effects of particles can be significant. As a result of a spatial expansion of the pair velocity distribution function about the contact point between two colliding particles, one can obtain the bulk viscosity by computing the rate of collisional transfer of linear momentum. The expression for μ_b so obtained is just the volumetric term given in (A 1). The interpretation of this term is rather obvious. It represents the resistance of finite gas particles towards deformation as a result of collisional transfer of linear momentum caused by either expansion or contraction of the gas. In the dilute concentration limits, this term vanishes, signifying the absence of the finite size effects of particles. This is consistent with the physics of dilute gases in which particles may be regarded as point masses and they pose little resistance in either expansion or contraction.

In the present study of slightly inelastic and slightly rough spheres, the rate of change of translational and rotational energies are each governed by the conservation equations (3.8) and (3.9). For example, the ratio of T_r to T_t given in (5.2) for the case of simple shear flow is derived from (3.9) and is plotted in figure 1. Since equipartition of energy is not realized in general owing to possible energy dissipation through inelastic and frictional collisions, the two terms involving T_t and T_r in the perturbation function ϕ which gave rise to the relaxation term in the bulk viscosity according to the theory of Theodosopulu & Dahler (1974*b*), Dahler & Theodosopulu (1975) are neglected in the present analysis. As a result, the present μ_b in (4.14) differs from (A 1) by the omission of the relaxation term. Nevertheless, (4.14) is consistent with the μ_b obtained in the kinetic theory of dense smooth hard-sphere gases, whereas (A 1) is not.

Theodosopulu & Dahler (1974*b*) obtained the following result for the collisional flux of translational energy:

$$\boldsymbol{q}_{t_c} = \frac{8}{5}\left(\frac{4+5K}{1+K}\right)\nu g_0 \boldsymbol{q}_{t_k} + \frac{4}{3}\left(\frac{2+3K}{1+K}\right)\nu g_0 \boldsymbol{q}_{r_k} - \frac{4}{\pi^{\frac{1}{2}}}\left(\frac{k}{m}\right)^{\frac{3}{2}}\left(\frac{1+2K}{1+K}\right)\rho\nu g_0 \sigma T_t^{\frac{1}{2}} \nabla T_t. \quad (A\ 4)$$

Note that we have neglected the kinetic flux of angular momentum, $\boldsymbol{L}_k$, term in (A 4) for the reason mentioned previously in §4. The omission of $\boldsymbol{L}_k$ would not affect any of the results discussed below. Taking $e = 1$ and $\beta = 1$ for the case of perfectly elastic and perfectly rough spheres, the present theory in (4.17) yields the following result for the collisional flux of translational energy:

$$\boldsymbol{q}_{t_c} = \frac{4}{5}\left(\frac{3+5K}{1+K}\right)\nu g_0 \boldsymbol{q}_{t_k} - \frac{4}{\pi^{\frac{1}{2}}}\left(\frac{1+2K}{1+K}\right)\rho\nu g_0 \sigma T_t^{\frac{1}{2}} \nabla T_t. \quad (A\ 5)$$

The result of Theodosopulu & Dahler in (A 4) differs from the present one in (A 5) in two respects. First, the $\boldsymbol{q}_{t_k}$ term (i.e. the first term) in (A 4) differs from that in (A 5) by an algebraic expression of parameter K in front. Secondly, (A 4) has an extra $\boldsymbol{q}_{r_k}$ term (i.e. the second term). To examine these discrepancies, we take $e = 1$ and $\beta = -1$ in (4.17) corresponding to the case of perfectly elastic and perfectly smooth spheres. The present result checks with the one obtained in the kinetic theory of smooth dense hard-sphere gases (Chapman & Cowling 1970) whereas (A 4) does not. This provides a verification not only for the numerical values in the algebraic expression of parameter K but also the independence of $\boldsymbol{q}_{r_k}$ as well. It seems possible that some numerical errors were made in the computations of Theodosopulu & Dahler.

REFERENCES

AHMADI, G. & SHAHINPOOR, M. 1983 A kinetic model for rapid flows of granular materials. *Intl J. Nonlin. Mech.* **19**, 177–186.

CAMPBELL, C. S. 1989 The stress tensor for simple shear flows of a granular material. *J. Fluid Mech.* **203**, 449–473.

CAMPBELL, C. S. 1990 Rapid granular flows. *Ann. Rev. Fluid Mech.* **22**, 57–92.

CAMPBELL, C. S. & BRENNEN, C. E. 1985 Computer simulation of granular shear flows. *J. Fluid Mech.* **151**, 167–188.

CAMPBELL, C. S. & GONG, A. 1986 The stress tensor in a two-dimensional granular shear flow. *J. Fluid Mech.* **164**, 107–125.

CHAPMAN, S. & COWLING, T. G. 1970 *The Mathematical Theory of Non-Uniform Gases*, 3rd edn. Cambridge University Press.

CONDIFF, D. W., LU, W. K. & DAHLER, J. S. 1965 Transport properties of polyatomic fluids, a dilute gas of perfectly rough spheres. *J. Chem. Phys.* **42**, 3445–3475.

CRAIG, K., BUCKHOLZ, R. H. & DOMOTO, G. 1986 An experimental study of the rapid flow of dry cohesionless metal powders. *Trans. ASME* E: *J. Appl. Mech.* **53**, 935–942.

DAHLER, J. S. & THEODOSOPULU, M. 1975 The kinetic theory of dense polyatomic fluids. *Adv. Chem. Phys.* **31**, 155–229.

DAVIS, H. T. 1973 Kinetic theory of dense fluids and liquids revisited. *Adv. Chem. Phys.* **24**, 257–343.

FARRELL, M., LUN, C. K. K. & SAVAGE, S. B. 1986 A simple kinetic theory for granular flow of binary mixtures of smooth, inelastic, spherical particles. *Acta Mech.* **63**, 45–60.

GOLDSMITH, W. 1960 *Impact: The Theory and Physical Behavior of Colliding Solids.* E. Arnold.

HAFF, P. K. 1983 Grain flow as a fluid-mechanical phenomenon. *J. Fluid Mech.* **134**, 401–430.

HANES, D. M. 1983 Studies on the mechanics of rapidly flowing granular-fluid materials. Ph.D. dissertation, University of California.

HANES, D. M. & INMAN, D. L. 1985 Observations of rapidly flowing granular-fluid materials. *J. Fluid Mech.* **150**, 357–380.

HOPKINS, M. A. & SHEN, H. H. 1986 Constitutive relations for a planar, simple shear flow of rough disks. *Intl J. Engng Sci.* **24**, 1717–1730.

HOPKINS, M. A. & SHEN, H. H. 1988 A Monte Carlo simulation of a rapid simple shear flow of granular materials. In *Micromechanics of Granular Materials* (ed. M. Satake & J. T. Jenkins), pp. 349–358. Elsevier.

JENKINS, J. T. 1987 Balance laws and constitutive relations for rapid flows of granular materials. In *Constitutive Models of Deformation* (ed. J. Chandra & R. Srivastav), pp. 109–199. SIAM.

JENKINS, J. T. & RICHMAN, M. W. 1985*a* Grad's 13-moment system for a dense gas of inelastic spheres. *Arch. Rat. Mech. Anal.* **87**, 355–377.

JENKINS, J. T. & RICHMAN, M. W. 1985*b* Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Phys. Fluids* **28**, 3485–3494.

JENKINS, J. T. & SAVAGE, S. B. 1983 A theory for the rapid flow of identical, smooth, nearly elastic particles. *J. Fluid Mech.* **130**, 187–202.

JOHNSON, K. L. 1982 One hundred years of Hertz contact. *Proc. Inst. Mech. Engrs* **196**, 363–378.

JOHNSON, K. L. 1985 *Contact Mechanics.* Cambridge University Press.

JOHNSON, P. C., NOTT, P. & JACKSON, R. 1990 Frictional–collisional equations of motion for particulate flows and their application to chutes. *J. Fluid Mech.* **210**, 501–535.

LUN, C. K. K. & SAVAGE, S. B. 1986 The effects of an impact dependent coefficient of restitution on stresses developed by sheared granular materials. *Acta Mech.* **63**, 15–44.

LUN, C. K. K. & SAVAGE, S. B. 1987 A simple kinetic theory for granular flow of rough, inelastic, spherical particles. *Trans. ASME* E: *J. Appl. Mech.* **54**, 47–53.

LUN, C. K. K., SAVAGE, S. B., JEFFREY, D. J. & CHEPURNIY, N. 1984 Kinetic theories for granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* **140**, 223–256.

MAW, N., BABER, J. R. & FAWCETT, J. N. 1976 The oblique impact of elastic spheres. *Wear* **38**, 101–114.

MAW, N., BABER, J. R. & FAWCETT, J. N. 1981 The role of elastic tangential compliance in oblique impact. *Trans. ASME* F: *J. Lubrication Technol.* **103**, 74–80.

McCOY, B. J., SANDLER, S. I. & DAHLER, J. S. 1966 Transport properties of polyatomic fluids. IV. The kinetic theory of a dense gas of perfectly rough spheres. *J. Chem. Phys.* **45**, 3485–3500.

McTIQUE, D. F. 1978 A model for stresses in shear flows of granular material. In *Proc. US–Japan. Semin. Continuum-Mechanics and Statistical Approaches to the Mechanics of Granular Material*, pp. 266–271. Tokyo: Gakujutsu Bunken Fukyukai.

NAKAGAWA, M. 1988 Kinetic theoretical approach for rapidly deforming granular material. In *Micromechanics of Granular Materials* (ed. M. Satake & J. T. Jenkins), pp. 331–339. Elsevier.

OGAWA, S., UMEMURA, A. & OSHIMA, N. 1980 On the equations of fully fluidized granular materials. *Z. Angew. Math. Phys.* **31**, 483–493.

RICHMAN, M. W. 1986 Kinetic theories for rate dependent granular flows. *Proc. 10th US Natl Congr. Appl. Mech., Austin, Texas* (ed. J. P. Lamb), pp. 339–346.

RICHMAN, M. W. 1989 The source of second moment in dilute granular flows of highly inelastic spheres. *J. Rheol.* **33**, 1293–1306.

SAVAGE, S. B. 1984 The mechanics of rapid granular flows. *Adv. Appl. Mech.* **24**, 289–366.

SAVAGE, S. B. 1991 Numerical simulations of Couette flow of granular materials; spatio-temporal coherence and $1/f$ noise. In *Physics of Granular Media* (ed. J. Dodds & D. Bideau). Nova Scientific Press.

SAVAGE, S. B. & JEFFREY, D. J. 1981 The stress tensor in a granular flow at high shear rates. *J. Fluid Mech.* **110**, 255–272.

SAVAGE, S. B. & SAYED, M. 1984 Stresses developed by dry cohesionless granular materials sheared in an annular shear cell. *J. Fluid Mech.* **142**, 391–430.

SHEN, H. H. & ACKERMANN, N. L. 1982 Constitutive relations for fluid-solids mixtures. *J. Eng. Mech. Div. ASCE* **108**, 748–763.

THEODOSOPULU, M. & DAHLER, J. S. 1974*a* Kinetic theory of polyatomic liquids. I. The generalized moment method. *J. Chem. Phys.* **60**, 3567–3582.

THEODOSOPULU, M. & DAHLER, J. S. 1974*b* The kinetic theory of polyatomic liquids. II. The rough sphere, rigid ellipsoid, and square-well ellipsoid models. *J. Chem. Phys.* **60**, 4048–4057.

WALTON, O. R. 1983 Particle dynamics calculations of shear flow. In *Mechanics of Granular Materials: New Model and Constitutive Relations* (ed. J. T. Jenkins & M. Satake), pp. 327–338. Elsevier.

WALTON, O. R. & BRAUN, R. L. 1986 Stress calculations for assemblies of inelastic spheres in uniform shear. *Acta Mech.* **63**, 73–86.

WERNER, B. T. & HAFF, P. K. 1988 The impact process in eolian saltation: two dimensional simulations. *Sedimentology* **35**, 267–278.